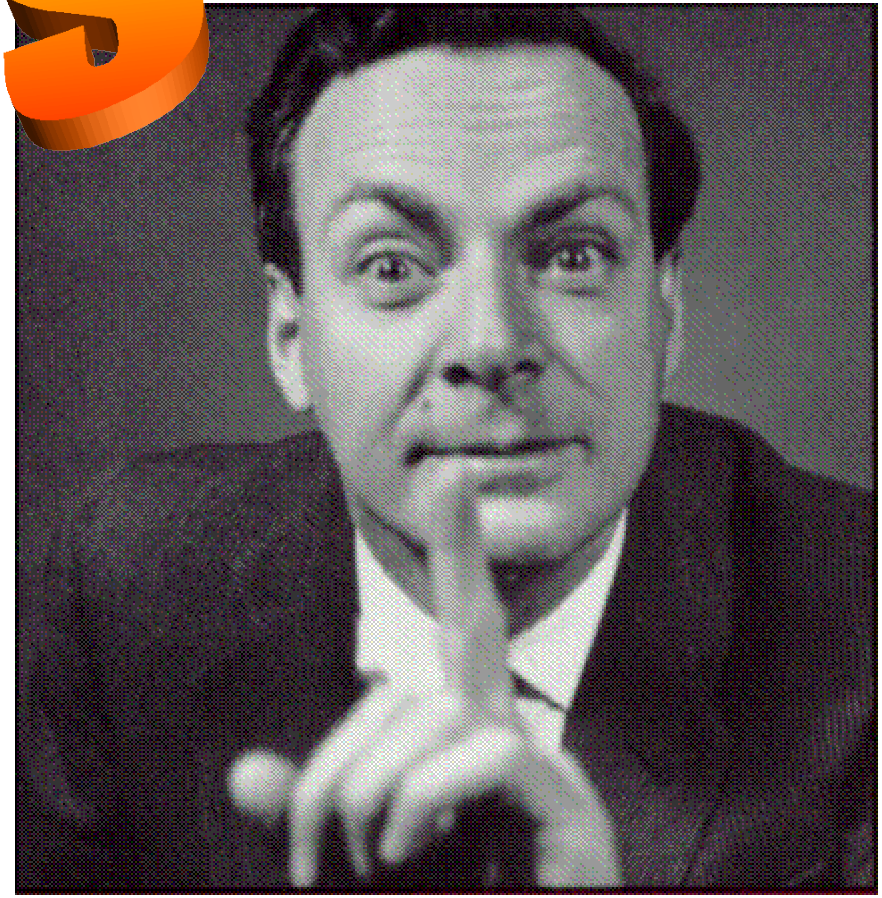
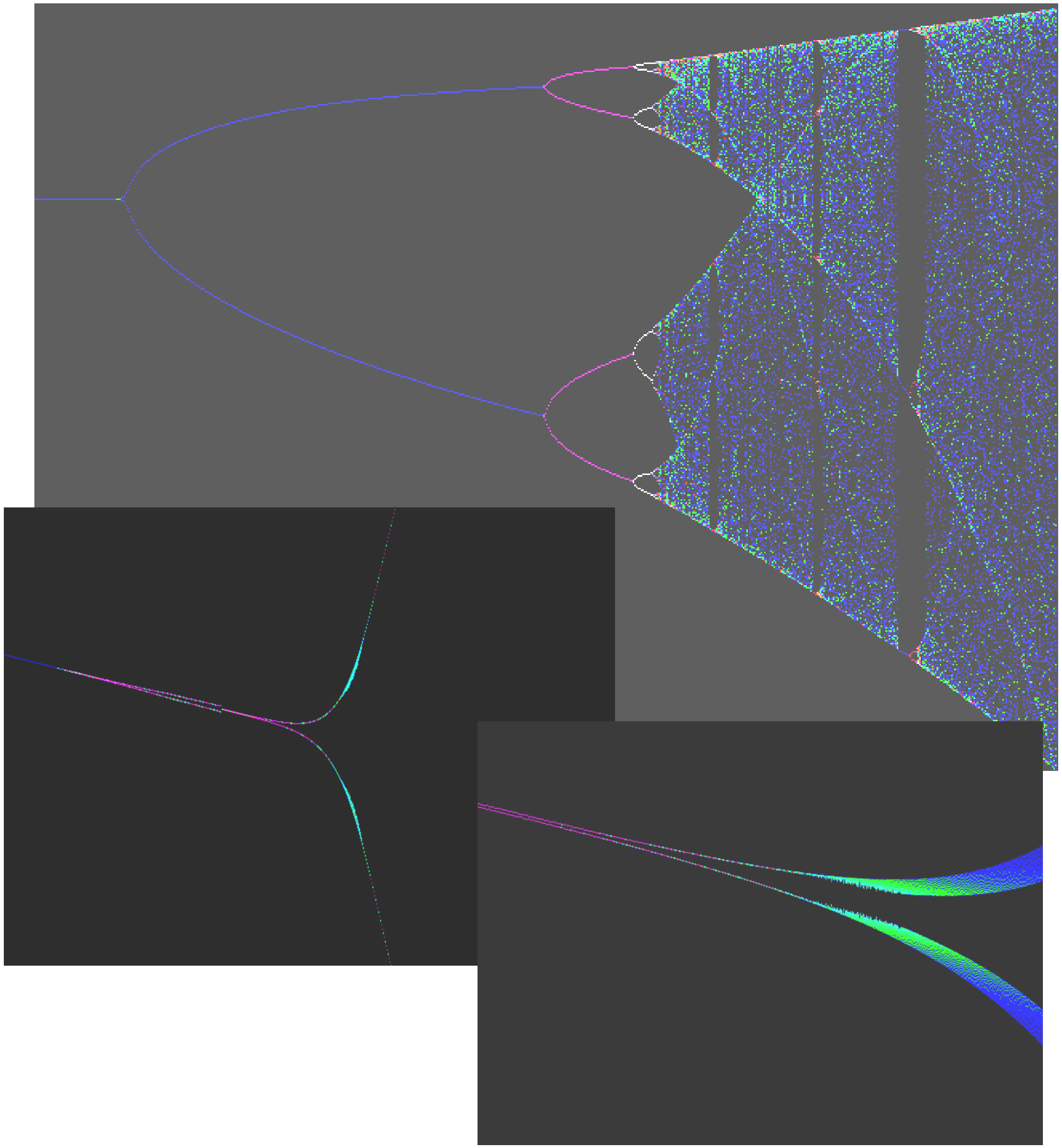
Chaos



Or, How I learned that Renormalization Group Theory has Nothing to do with Group Theory.

Research conducted by Brad Wogsland working under Prof. Wiesenfeld.

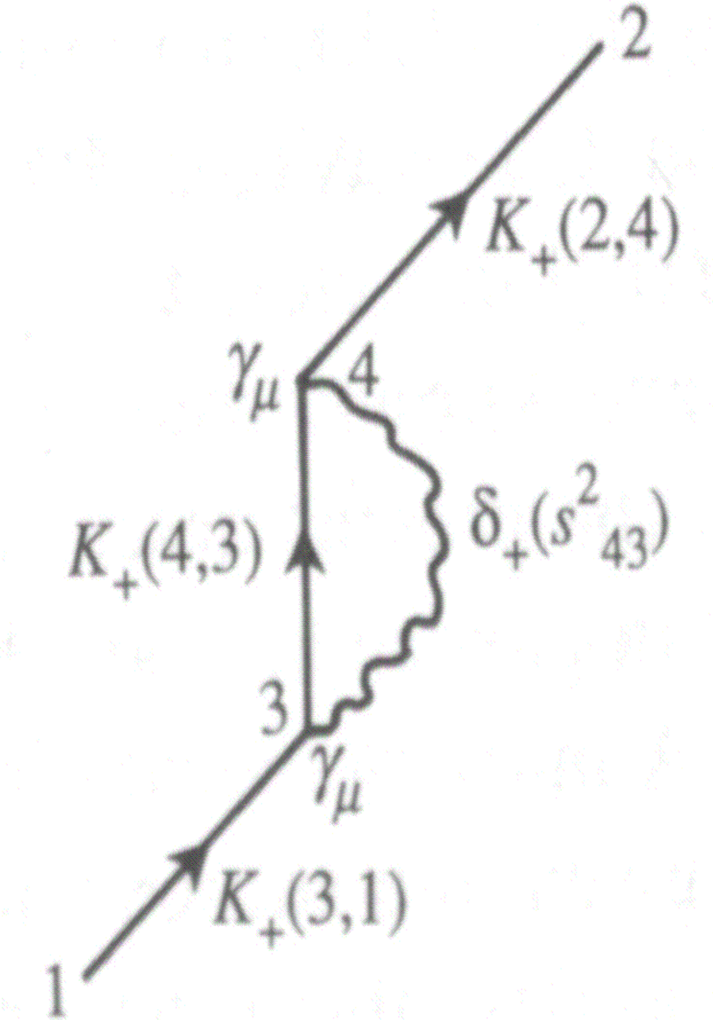
I. Motivation - the Feigenbaum Constant (□ = 4.669202 . . .)



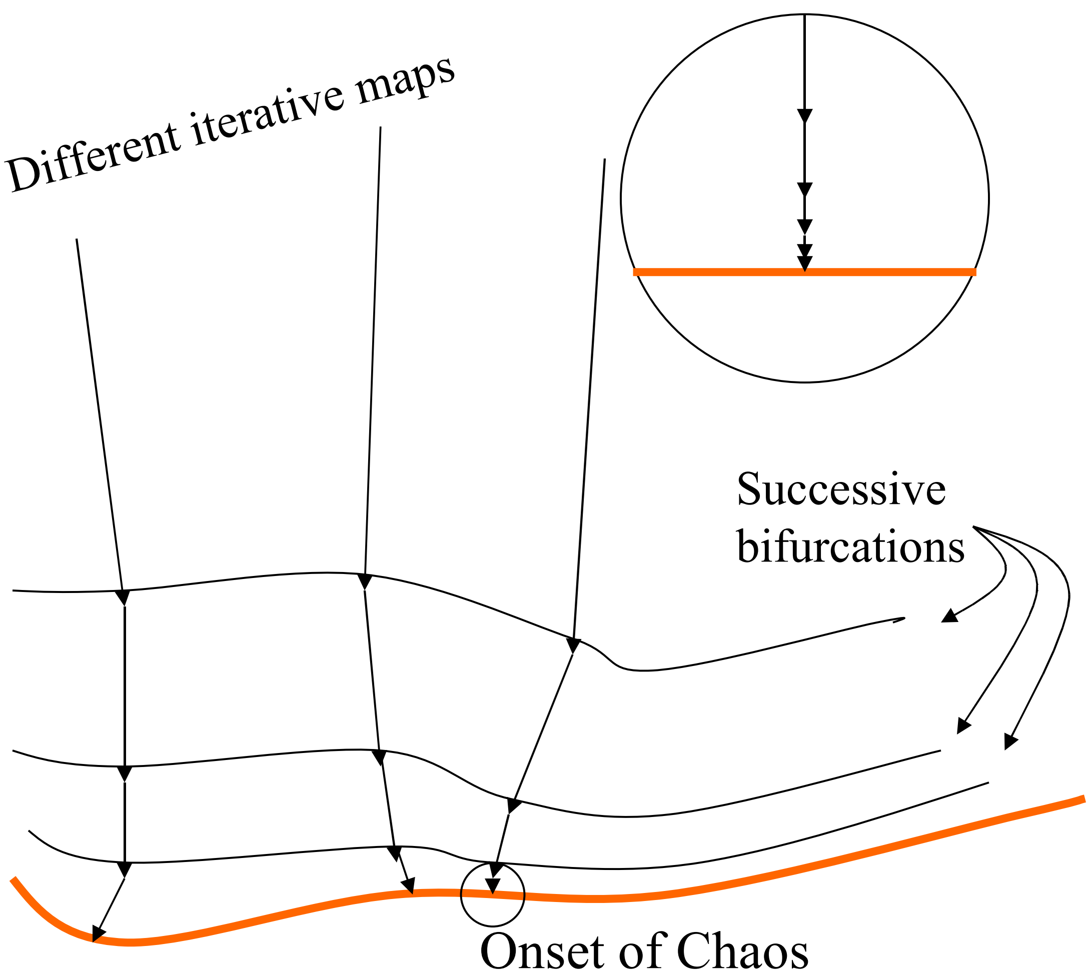
- $x_{n+1} = f(a, x_n)$
- Examples: $x_{n+1} = a\, x_n(1 - x_n)$,
  $x_{n+1} = a \sin(\alpha\, x_n)$, etc.
- Bifurcation diagram (x_n vs a)
- Feigenbaum constant is the limit of ratios of distances between bifurcations - same number for all maps of this form!
- Close-ups of the first bifurcation pt show that, numerically, the precise location of these pts is hard to find, making calculation of the constant to increasing precision difficult
- Is there another way to calculate this constant?

II. Enter the Renormalization Group

- First appearing in the mid-1900s as a way to deal with infinities arising in Quantum Electrodynamical (QED) calculation of the electron's self energy among other things.
- Recall the electric potential is U~1/r, so if r = 0 this would yield infinity.
- Current theoretical calculations using these methods give results more precise than the 11 significant figures found by experiment.
- Has since been applied to many similar problems, including critical phenomena and the onset of chaos.
- This body of techniques uses scaling laws to reach an approximate value, and in the limit, the correct value for attributes of these problems.



III. Renormalization Group Theory



- Graph of different iterative maps in the function space of all unimodal functions.
- All maps intersect the chaotic boundary orthogonally (inset).
- Thus, different maps all look the same in the limit as they approach chaos - “Universality”.
- This is the reason that the Feigenbaum constant is the same for all of these maps.
- So if we make a few simplifications, the problem becomes much easier to work with...

IV. Calculation

- Instead consider the simpler function space of functions of the form

  $f(x) = a - bx^2 + cx^4$
- Applying the transformation

  $T[f(x)] = \alpha f(f(x/\alpha))$
- yields the nonlinear map

  $a` = a\ \alpha(1 - ab + a^3 c)$

  $b` = -(1/\alpha)(2ab(b - 2a^2 c))$

  $c` = -(1/\alpha^3)(b^3 + 2abc - 6a^2 b^2 c - 4a^3 c^2)$
- which has fixed points only if

  $\alpha^3 - \alpha - 24\alpha^2 + 32\alpha^3/(\beta + 4\alpha) = 0$

  where $\beta = 1 - 1/\alpha + \alpha/4$.
- By the way, $\alpha$is called the universal scaling constant, which we calculate here to be $\alpha = -2.53403$. Not far off from the real value of $\alpha = -2.5029$….
- And if we calculate the eigenvalues of the fixed pts of the map above...
- The largest of these eigenvalues yields $\delta = 4.7425$ Again, this is not far off from the accepted value of 4.6992...
- Because we are dealing with these very nicely behaved power series, this approximation method is tractable,
- i.e. although the method at first seems more difficult, the difficulty increases polynomially with increasing accuracy as opposed to the exponentially increasing difficulty of the other method.

IV. Future Avenues to Explore

- What about other universality classes (i.e. functions without a quadratic maximum at $x = 0$)?
- Are there any universality classes which don't yield to these methods?
- Quantitatively, how good is this method? And, is there a simpler tractable method for calculating δ?
- Any other questions?

V. References (In no particular order)

- Conversations with Prof. Wiesenfeld
- Manfred Schroeder, *Fractal, Chaos, Power Laws: Minutes from an Infinite Paradise*, (1991).
- Pictures created with the free software Fractint v19.6, available at various locations on the internet.
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