The RLC Circuit

A circuit with a resistor, an inductor, and a capacitor connected in series, commonly called an RLC circuit, is described by the following differential equation,

$$LQ''+RQ'+Q/C = V(t) \tag{1}$$

where Q is the charge (Q' being the current), L is the inductance of the inductor, R is the resistance of the resistor, C is the capacitance of the capacitor, and V(t) is a time dependent driving voltage. Note the striking similarity to the damped spring differential equation,

$$mx''+cx'+kx = F(t) \tag{1'}$$

where x is the position, m is the mass, c is the damping constant, k is the spring constant, and F(t) is a time dependent applied force. Remember the same equations have the same solutions, regardless of what you call the variables involved.

Assuming a sinusoidal driving voltage of the form $V(t) = V_0 \sin\omega t$ yields

$$LQ''+RQ'+Q/C = V_0 \sin\omega t \tag{2}$$

and differentiating

$$LI''+RI'+I/C = \omega V_0 \cos\omega t \tag{3}$$

where I, of course, is the current. The solution to (3) will be a sum of two parts, the homogeneous solution $I_h(t)$ and the non-homogeneous solution $I_{osc}(t)$. Solving the characteristic equation

$$Lr^2+Rr+1/C = 0 \tag{4}$$

yields roots

$$r = \frac{-R \pm \sqrt{R^2-4L/C}}{2L} \tag{5}$$

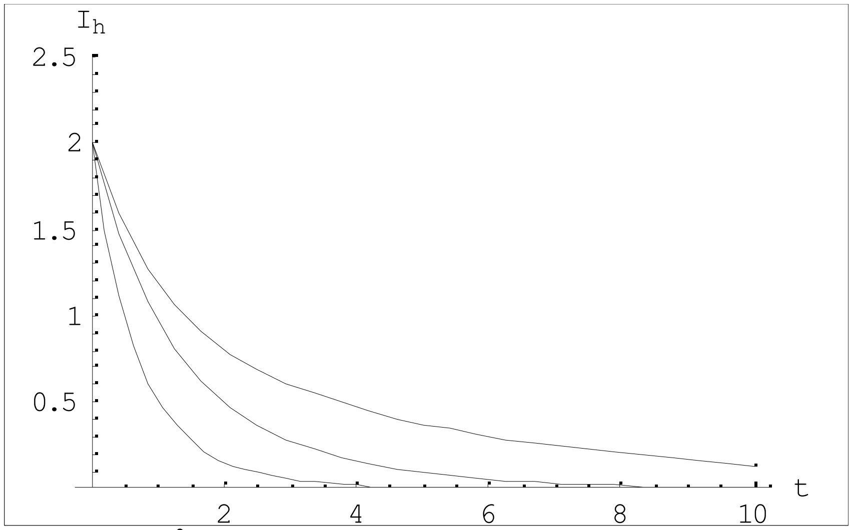
which leads to three cases (the nomenclature is in analogy to the damped spring):

1. **Overdamped** ($R^2 > 4L/C$)

For this case the characteristic equation has two distinct negative real roots $-\alpha$ and $-\beta$ yielding the homogeneous solution

$$I_h(t) = c_1 e^{-\alpha t} + c_2 e^{-\beta t} \tag{6}$$

where c_1 and c_2 are arbitrary constants. Below are several plots of this solution for various values of the constants.

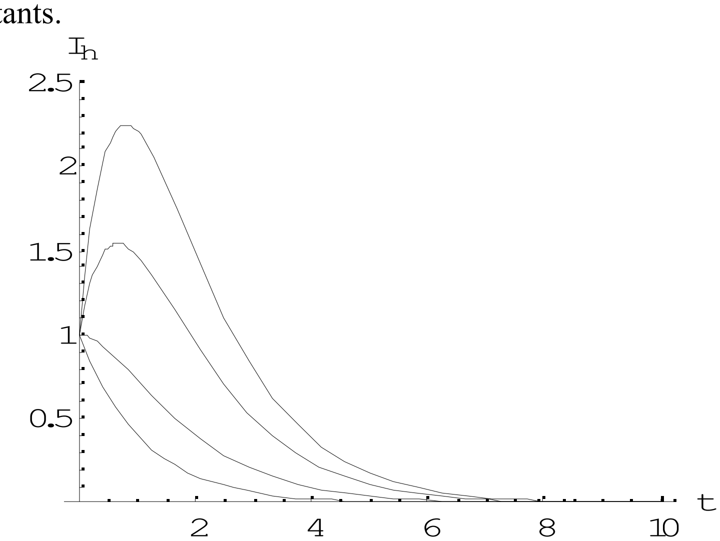


2. **Critically damped** ($R^2 = 4L/C$)

For this case the characteristic equation has a single distinct negative real root -r yielding the homogeneous solution

$$I_h(t) = e^{-rt}(c_1 + c_2 t) \tag{7}$$

where c_1 and c_2 are arbitrary constants. Below are four plots of this solution for different choices of constants.

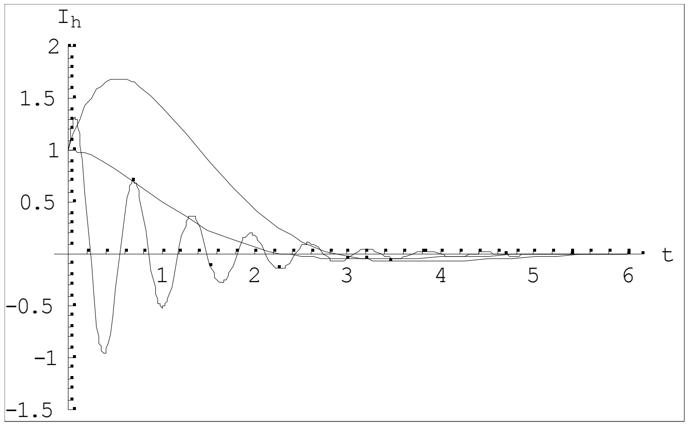


3. **Underdamped** ($R^2 < 4L/C$)

For this case the characteristic equation has two distinct complex roots $-r \pm \gamma i$ yielding the homogeneous solution

$$I_h(t) = e^{-rt}(c_1 \cos \gamma t + c_2 \sin \gamma t) \tag{8}$$

where c_1 and c_2 are arbitrary constants. Below are several plots of the solution with various constants.



Looking at these three possibilities it is clear that $I_h(t)$ will die off in any case leaving $I(t) \approx I_{osc}(t)$ after some finite time. Using the method of undetermined coefficients we find the non-homogeneous solution to be

$$I_{osc}(t) = I_0\cos(\omega t-\varphi) \tag{9}$$

where $I_0 = V_0/\sqrt{R^2 + (\omega L - 1/\omega C)^2}$ and $\varphi = \tan^{-1}\omega RC/(1-LC\omega^2)$. The quantity in the denominator

$$Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2} \tag{10}$$

is commonly called the impedance. Note that the amplitude of $I_{osc}(t)$ is

$$I_0 = V_0/Z \tag{11}$$

which should remind you of Ohm's Law, with Z as a coefficient that describes how a given circuit *impedes* the flow of current. It should not be surprising that the units of Z are Ohms [Ω]. Furthermore, there is also a special name for the the quantity in parentheses,

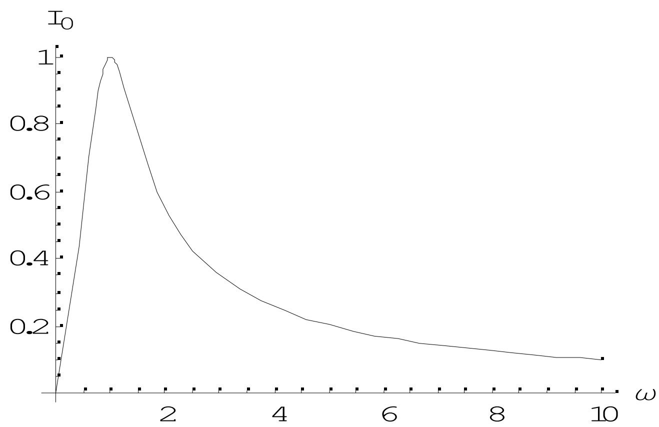
$$S = \omega L - 1/\omega C \tag{12}$$

the reactance, which also has units [Ω]. This quantity tells you how a given circuit *reacts* to a certain frequency driving voltage.

Now if we look at I_0 as a function of ω (pictured below), it is clear that to maximize this amplitude we must have S = 0, or

$$\omega_R = (LC)^{-½} \tag{13}$$

so that Z = R. This frequency, ω_R, is called the resonant frequency.



References:

Edwards, C.H. Jr. and David E. Penney, *Differential Equations: Computing and Modeling*, Ch. 3 (1996).

The graphs were produced using *Mathematica*.